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Harmonic oscillator with exponentially decaying mass

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Abstract. The problem of a harmonic oscillator with varying mass parameter is reduced by canonical transformation to the corresponding constant mass problem and is solved in the case of an exponentially decaying mass. It is shown that the constructed canonical Hamiltonian has time-independent eigenvalues and eigenvectors. The cases of undercritical and overcritical damping are considered in detail. The Green function is calculated and the behaviour of coherent states is discussed.

The theory is related to the case of a cavity oscillator with a decaying field as in threshold laser operation. In particular, the energy of the field is considered.

1. Introduction

Much attention has been paid in recent years to time-dependent quantal systems and especially to dissipative systems (Hasse 1975, 1978, Tartaglia 1977, Dodonov and Man'ko 1979, Agayeva 1980, Remaud and Hernandez 1980, Stevens 1980). A solution of the problem of a variable mass, or variable frequency, harmonic oscillator would be welcomed in several branches of physics and considerable effort has been devoted to finding invariants of the motion for such a system (Lewis and Riesenfeld 1969, Sarlet 1978, Dodonov and Man'ko 1979).

The treatment of a decaying oscillator to be presented in this paper originated in an interest in a cavity oscillator in which the electromagnetic field varies with time under the action of some reservoir as, for instance, in laser production. The theory that we shall develop is applicable to any physical situation involving damped oscillator motion, but it is useful to have a definite problem in mind to help fix our ideas, especially in relation to the energy of the system. Thus we shall begin by briefly considering a cavity oscillator when the field is subjected to some external influence, particularly to a decay.

The quantisation of the field in a cavity is described by Sargent *et al* (1974). The energy of the field

$$E=\tfrac{1}{2}\int_V \mathrm{d}\tau\,(\varepsilon_0\mathscr{E}_x^2+\mu_0\mathscr{H}_y^2) \tag{1.1}$$

is quantised to give

$$E\to H(q,p,t)=\tfrac{1}{2}p^2/M(t)+\tfrac{1}{2}M(t)\omega_0^2q^2, \qquad [q,p]=\mathrm{i}\hbar. \tag{1.2}$$

The mass is taken as a function of time to simulate externally imposed changes in the field, as discussed by Colegrave and Abdalla (1981). For an exponential decay in the cavity we take

$$M(t)=M_0\exp(-2\gamma t), \qquad \gamma>0. \tag{1.3}$$

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Then, since both $\mathscr{E}_x^2$ and $\mathscr{H}_y^2$ are proportional to $M(t)$, we see from equation (1.1) that the energy of the cavity field is given by

$$E = \exp(-2\gamma t)E_0, \tag{1.4}$$

where E_0 denotes the energy when $M = M_0$. In quantised form

$$E_0 \to H(Q, P),$$

where

$$H(Q, P) = \tfrac{1}{2}P^2/M_0 + \tfrac{1}{2}M_0\omega_0^2 Q^2, \qquad [Q, P] = i\hbar. \tag{1.5}$$

Combining (1.4) and (1.5), we may define an energy operator

$$E(Q, P, t) = \exp(-2\gamma t)H(Q, P). \tag{1.6}$$

Hasse (1975) and Tartaglia (1977) have taken a relation similar to (1.6) to connect the energy and the Hamiltonian of a damped oscillator, but it should be noted that their H, unlike ours, is explicitly time-dependent. From the relation (1.4) we see the problem quite clearly as one of decay, whereas in (1.2) the Hamiltonian consists of two parts, say $T + V$, where from (1.3) we see that T increases and V decreases with time (or vice versa if we change the sign of γ). Thus if we start with equations (1.2) and (1.3) to describe the system it is not so clear that $\gamma > 0$ is necessary for decay. In fact, most authors work with $\gamma < 0$ in (1.3) and, as we shall discuss later, this does not affect the decaying oscillator problem when treated in isolation.

A quantal treatment of the decay problem based on the Hamiltonian (1.2) together with (1.3) (but without the requirement $\gamma > 0$) has been considered already by many authors (see e.g. Hasse 1975, Tartaglia 1977, Dodonov and Man'ko 1979). Our approach differs from theirs in that we shall change the system (1.2) to the simple time-independent system (1.5) by a time-dependent canonical transformation, to be discussed in the next section. Some similarities will be found to the results obtained by the above-mentioned authors, but some important differences will be apparent.

After introducing the canonical transformation in § 2, we consider the dynamics of the decaying oscillator firstly in the wave picture, then in the Heisenberg picture and finally we adopt the direct approach of Dirac. We shall calculate the Green function and consider the existence of coherent states. Finally we shall discuss the energy of a damped cavity oscillator.

2. The time-dependent canonical transformation

The scaling transformation

$$Q(t) = [M(t)/M_0]^{1/2}q, \qquad P(t) = [M_0/M(t)]^{1/2}p, \tag{2.1}$$

makes $H(q, p, t)$ of equation (1.2) equal to $H(Q, P)$ of equation (1.5). This is a canonical transformation with generating function in the notation of Goldstein (1950) given by

$$F_2(q, P, t) = [M(t)/M_0]^{1/2}Pq. \tag{2.2}$$

The new canonical Hamiltonian is

$$K(Q, P, t) = H + \partial F_2/\partial t. \tag{2.3}$$

Using equations (2.1), (2.2) and (2.3) we obtain

$$K=\frac{1}{2M_0}P^2+\tfrac{1}{2}M_0\omega_0^2Q^2+\frac{1}{4M}\frac{\mathrm{d}M}{\mathrm{d}t}(QP+PQ), \tag{2.4}$$

where we have written PQ in the self-adjoint form $\frac{1}{2}(QP+PQ)$. For a decaying oscillator we choose $M(t)$ according to equation (1.3); then equation (2.4) becomes

$$K=\tfrac{1}{2}P^2/M_0+\tfrac{1}{2}M_0\omega_0^2Q^2-\tfrac{1}{2}\gamma(QP+PQ). \tag{2.5}$$

We note that the new Hamiltonian is independent of the time. This fortuitous elimination of the time enables us to treat the original time-dependent problem (1.2) subject to (1.3) by ordinary time-independent quantum mechanics. In the next section we treat the problem by wave mechanics.

3. The eigenvalues and eigenfunctions of *K*

We shall for the moment restrict ourselves to the case of undercritical damping. Overcritical damping, when $\gamma>\omega_0$, will be considered in § 4.2 and in the subsequent sections. The main purpose of the present section is comparison with the work of other authors, notably of Hasse (1975) and of Tartaglia (1977).

We seek eigenvalues λ_l and eigenvectors $|l\rangle$ of the canonical Hamiltonian K given by equation (2.5); thus

$$K|l\rangle=\lambda_l|l\rangle. \tag{3.1}$$

The corresponding Schrödinger equation is

$$\left[-\frac{\hbar^2}{2M_0}\frac{\mathrm{d}^2}{\mathrm{d}Q^2}+\tfrac{1}{2}M_0\omega_0^2Q^2+\mathrm{i}\hbar\gamma\left(Q\frac{\mathrm{d}}{\mathrm{d}Q}+\tfrac{1}{2}\right)\right]\psi_l(Q)=\lambda_l\psi_l(Q). \tag{3.2}$$

The reduced frequency ω is defined by

$$\omega^2=\omega_0^2-\gamma^2. \tag{3.3}$$

Then the eigenvalues are easily found to be

$$\lambda_l=\hbar\omega(l+\tfrac{1}{2}), \qquad l=0,1,2,\ldots \tag{3.4}$$

and the normalised eigenfunctions are

$$\psi_l(Q)=N_l\exp[-M_0(\omega-\mathrm{i}\gamma)Q^2/(2\hbar)]H_l(\sqrt{\alpha}Q), \tag{3.5}$$

where

$$N_l=(\alpha/\pi)^{1/4}(2^l l!)^{-1/2}, \qquad \alpha=M_0\omega/\hbar. \tag{3.6}$$

The corresponding solution of the time-dependent Schrödinger equation $K\psi=\mathrm{i}\hbar\partial\psi/\partial t$ is

$$\psi_l(Q,t)=\psi_l(Q)\exp[-\mathrm{i}(l+\tfrac{1}{2})\omega t]. \tag{3.7}$$

We notice that the second part of K, coming from $\partial F_2/\partial t$, gives rise to the phase factor $\exp[\mathrm{i}M_0\gamma Q^2/(2\hbar)]$ in (3.5). Remembering the connection (2.1) between Q and the original coordinate q, we see that our solutions (3.7) agree with the solutions of Hasse (1975) and of Tartaglia (1977) except that their solutions contain a factor $\exp(-\frac{1}{2}\gamma t)$ (in our notation). This follows from (3.5) on normalising with respect to q rather than Q.

To calculate the matrix elements of the original Hamiltonian operator $H = K - \mathrm{i}\gamma\hbar(Q\,\mathrm{d}/\mathrm{d}Q + \frac{1}{2})$, related to the energy E by equation (1.6), is straightforward. We find that

$$\langle l|H|l'\rangle = (l+\tfrac{1}{2})\hbar(\omega_0^2/\omega) - \tfrac{1}{2}\mathrm{i}\gamma\hbar[l'(l'-1)]^{1/2}(1+\mathrm{i}\gamma/\omega)\delta_{l,l'-2} + \tfrac{1}{2}\mathrm{i}\gamma\hbar[l(l-1)]^{1/2}(1-\mathrm{i}\gamma/\omega)\delta_{l,l'+2}. \tag{3.8}$$

Thus, in the eigenstate $|l\rangle$ of K, the expectation value of H is

$$\langle H\rangle = (l+\tfrac{1}{2})\hbar\omega_0^2/\omega, \tag{3.9}$$

which agrees with the result obtained by Hasse (1975) and Tartaglia (1977).

Clearly a unitary matrix U exists that will transform the matrix H, given by (3.8), to diagonal form: $H_\mathrm{D} \to H = \tilde{U}HU$, where

$$\langle n|H_\mathrm{D}|n'\rangle = (n+\tfrac{1}{2})\hbar\omega_0\delta_{nn'}, \qquad n = 0, 1, 2, \ldots. \tag{3.10}$$

We shall give the explicit form for this transformation in § 7.

4. Equations of motion

We shall use the Heisenberg equations with respect to the canonical Hamiltonian K. Thus for any operator $\mathcal{O}$,

$$d\mathcal{O}/\mathrm{d}t = \partial\mathcal{O}/\partial t + (\mathrm{i}\hbar)^{-1}[\mathcal{O}, K]. \tag{4.1}$$

For the coordinate and momentum this leads to the coupled equations

$$\mathrm{d}Q/\mathrm{d}t + \gamma Q = P/M_0, \qquad \mathrm{d}P/\mathrm{d}t - \gamma P = -M_0\omega_0^2 Q. \tag{4.2}$$

4.1. Case of undercritical damping

Firstly we consider the case when the mass decay rate is below critical, so that $\omega^2 = \omega_0^2 - \gamma^2 > 0$. Then the solutions of equations (4.2) are oscillatory:

$$Q(t) = [\cos\omega t - (\gamma/\omega)\sin\omega t]Q(0) + (\omega M_0)^{-1}\sin(\omega t)P(0), \tag{4.3a}$$

$$P(t) = [\cos\omega t + (\gamma/\omega)\sin\omega t]P(0) - (M_0\omega_0^2/\omega)\sin(\omega t)Q(0). \tag{4.3b}$$

It is interesting to compare these and subsequent results in this section with those of Hasse (1975, 1978) and of Dodonov and Man'ko (1979), but we note that for this γ must be changed in sign (and halved for comparison with the work of Hasse).

We write equation (2.5) in the form

$$K = T + V - \mathrm{i}\gamma[T, V]/(\hbar\omega_0^2), \tag{4.4}$$

where $T = P^2/(2M_0)$, $V = \frac{1}{2}M_0\omega_0^2Q^2$. Then from equation (4.1)

$$\mathrm{d}T/\mathrm{d}t - 2\gamma T = -\omega_0^2(QP+PQ)/2, \tag{4.5a}$$

$$\mathrm{d}V/\mathrm{d}t + 2\gamma V = \omega_0^2(QP+PQ)/2. \tag{4.5b}$$

Eliminating $QP + PQ$, we obtain

$$\mathrm{d}^2(T+V)/\mathrm{d}t^2 + 4\omega^2(T+V) = 4\omega_0^2 K. \tag{4.6}$$

Remembering that K is constant in time, we may integrate equation (4.6) to obtain

$$H = T + V = (\omega_0/\omega)^2 K + \tfrac{1}{2}(\alpha + \alpha^\dagger)\cos 2\omega t - \tfrac{1}{2}\mathrm{i}(\alpha - \alpha^\dagger)\sin 2\omega t \tag{4.7}$$

where

$$\alpha = (1 + \mathrm{i}\gamma/\omega)T(0) + (1 - \mathrm{i}\gamma/\omega)V(0) - (\omega_0/\omega)^2 K. \tag{4.8}$$

Equations (4.5) yield

$$\begin{aligned} QP + PQ = {} & (2\gamma/\omega^2)K + (2/\omega)[T(0) - V(0)]\sin 2\omega t \\ & + (2/\gamma)[T(0) + V(0) - (\omega_0/\omega)^2 K]\cos 2\omega t, \end{aligned} \tag{4.9}$$

$$\begin{aligned} T = {} & \tfrac{1}{2}(\omega_0/\omega)^2[1 - \cos 2\omega t + (\omega/\gamma)\sin 2\omega t]K \\ & + \{\cos 2\omega t - [(\omega^2 - \gamma^2)/(2\gamma\omega)]\sin 2\omega t\}T(0) \\ & - \tfrac{1}{2}(\omega_0^2/\gamma\omega)\sin(2\omega t)V(0), \end{aligned} \tag{4.10}$$

$$\begin{aligned} V = {} & \tfrac{1}{2}(\omega_0/\omega)^2[1 - \cos 2\omega t - (\omega/\gamma)\sin 2\omega t]K \\ & + \{\cos 2\omega t + [(\omega^2 - \gamma^2)/(2\gamma\omega)]\sin 2\omega t\}V(0) + \tfrac{1}{2}(\omega_0^2/\gamma\omega)\sin(2\omega t)T(0). \end{aligned} \tag{4.11}$$

4.2. Case of overcritical damping

We may extend our results to the case when the mass decay rate is so fast that $\omega^2 < 0$. In this case we define

$$\eta^2 = \gamma^2 - \omega_0^2, \tag{4.12}$$

and put $\omega \to \mathrm{i}\eta$. For instance, for $H = T + V$ we have from equations (4.10) and (4.11), when the damping is overcritical,

$$H(t) = (\omega_0/\eta)^2(\cosh 2\eta t - 1)K + [(1 + \gamma/\eta)T(0) + (1 - \gamma/\eta)V(0)]\cosh 2\eta t. \tag{4.13}$$

For large values of t we find, with $\eta > 0$,

$$T(t) \sim \tfrac{1}{4}\mathrm{e}^{2\eta t}\left[\left(\frac{\omega_0}{\eta}\right)^2\left(1 + \frac{\eta}{\gamma}\right)K + \frac{(\eta + \gamma)^2}{\eta\gamma}T(0) - \frac{\omega_0^2}{\eta\gamma}V(0)\right], \tag{4.14}$$

$$V(t) \sim \tfrac{1}{4}\mathrm{e}^{2\eta t}\left[\left(\frac{\omega_0}{\eta}\right)^2\left(1 - \frac{\eta}{\gamma}\right)K - \frac{(\eta - \gamma)^2}{\eta\gamma}V(0) + \frac{\omega_0^2}{\eta\gamma}T(0)\right], \tag{4.15}$$

$$E(t) \sim \tfrac{1}{2}\mathrm{e}^{-2(\gamma - \eta)t}\left[\left(\frac{\omega_0}{\eta}\right)^2 K + \left(1 + \frac{\gamma}{\eta}\right)T(0) + \left(1 - \frac{\gamma}{\eta}\right)V(0)\right]. \tag{4.16}$$

5. Dirac formalism

To continue our solution of the decaying oscillator problem, it is convenient to employ Dirac notation and introduce

$$A(t) = (2M_0\hbar\omega_0)^{-1/2}[M_0\omega_0 Q(t) + \mathrm{i}P(t)] \tag{5.1}$$

and its adjoint $A^\dagger(t)$. Then at all times

$$[Q, P] = \mathrm{i}\hbar \Rightarrow [A, A^\dagger] = 1, \tag{5.2}$$

and equation (2.5) becomes

$$K = \hbar\omega_0(A^\dagger A + \tfrac{1}{2}) - \tfrac{1}{2}\mathrm{i}\hbar\gamma(A^{\dagger 2} - A^2), \tag{5.3}$$

where $\mathrm{d}K/\mathrm{d}t = \partial K/\partial t = 0$.

We see from the definition (5.1) that A depends on t only through Q and P; hence $\partial A/\partial t = 0$ and equation (4.1) gives

$$\mathrm{d}A/\mathrm{d}t = -\mathrm{i}\omega_0 A - \gamma A^\dagger. \tag{5.4}$$

Again, let us begin by considering the case of light damping. On combining equation (5.4) with its adjoint we obtain

$$\mathrm{d}^2A/\mathrm{d}t^2 + \omega^2 A = 0, \tag{5.5}$$

and the solution is

$$A(t) = [\cos\omega t - \mathrm{i}(\omega_0/\omega)\sin\omega t]A(0) - (\gamma/\omega)\sin\omega t A^\dagger(0). \tag{5.6a}$$

For the heavy-damping case, $\eta^2 = \gamma^2 - \omega_0^2 > 0$, we make the analytic continuation $\omega \to \mathrm{i}\eta$ ($\eta > 0$) to obtain

$$A(t) = [\cosh\eta t - \mathrm{i}(\omega_0/\eta)\sinh\eta t]A(0) - (\gamma/\eta)\sinh(\eta t)A^\dagger(0) \tag{5.6b}$$

However, rather than work in the Heisenberg representation, we shall as far as possible use the time-free representation described by equations (5.2) and (5.3) or representations derived from this by canonical transformation as in the next section.

6. Diagonalising transformations

The original Hamiltonian H (and hence the energy E) is most simply treated in the time-free representation described by equations (1.5) or, equivalently,

$$H = \hbar\omega_0(A^\dagger A + \tfrac{1}{2}), \qquad [A, A^\dagger] = 1; \tag{6.1}$$

but, to continue with our analysis and compare with the work of other authors, we shall need to change to a representation in which K is diagonal.

6.1. Case of undercritical damping

The canonical transformation

$$\begin{pmatrix} A \\ A^\dagger \end{pmatrix} \to \begin{pmatrix} B \\ B^\dagger \end{pmatrix} = \begin{pmatrix} \cosh u & -\mathrm{i}\sinh u \\ \mathrm{i}\sinh u & \cosh u \end{pmatrix}\begin{pmatrix} A \\ A^\dagger \end{pmatrix}, \tag{6.2}$$

with

$$\cosh u = [(\omega_0 + \omega)/(2\omega)]^{1/2}, \qquad \sinh u = [(\omega_0 - \omega)/(2\omega)]^{1/2}, \tag{6.3}$$

causes the canonical Hamiltonian K given by equation (5.3) to assume a form which could be anticipated from the eigenvalues given by equation (3.4):

$$K \to K' = \hbar\omega(B^\dagger B + \tfrac{1}{2}), \qquad [B, B^\dagger] = 1. \tag{6.4}$$

The original Hamiltonian $H = \hbar\omega_0(A^\dagger A + \frac{1}{2})$ transforms to

$$H = \hbar(\omega_0^2/\omega)(B^\dagger B + \tfrac{1}{2}) + \tfrac{1}{2}\mathrm{i}\hbar\gamma(\omega_0/\omega)(B^{\dagger 2} - B^2). \tag{6.5}$$

Again, we have the expectation value of Hasse (1975) and of Targalia (1977) given by equation (3.9).

From equations (5.6*a*) and (6.2), or by solving the Heisenberg equation of motion for B, we find that

$$B(t)=[A(0)\cosh u-\mathrm{i}A^{\dagger}(0)\sinh u]\exp(-\mathrm{i}\omega t), \tag{6.6}$$

from which it may be checked that $B^{\dagger}B$, and hence K, is time-independent. The original Hamiltonian H, on the other hand, is dependent on the time. This reflects the fact that H and K do not commute, hence H cannot be a constant of the motion.

It is interesting to apply the transformation (6.2) a second time; thus

$$\begin{pmatrix}\bar{B}\\ \bar{B}^{\dagger}\end{pmatrix}=\begin{pmatrix}\cosh 2u & -\mathrm{i}\sinh 2u\\ \mathrm{i}\sinh 2u & \cosh 2u\end{pmatrix}\begin{pmatrix}A\\ A^{\dagger}\end{pmatrix}=\frac{1}{\omega}\begin{pmatrix}\omega_0 & -\mathrm{i}\gamma\\ \mathrm{i}\gamma & \omega_0\end{pmatrix}\begin{pmatrix}A\\ A^{\dagger}\end{pmatrix}. \tag{6.7}$$

We then find

$$K\to\bar{K}(\gamma\to-\gamma)=\hbar\omega_0(\bar{B}^{\dagger}\bar{B}+\tfrac{1}{2})+\tfrac{1}{2}\mathrm{i}\hbar\gamma(\bar{B}^{\dagger 2}-\bar{B}^2),\qquad [\bar{B},\bar{B}^{\dagger}]=1. \tag{6.8}$$

Comparing with equation (5.3), we see that the only difference between the representations is the sign of γ. This confirms a point that we mentioned in the Introduction: the system is invariant with respect to the symmetry $\gamma\to-\gamma$.

6.2. Case of overcritical damping

The spectrum described by equation (3.4) becomes continuous as $\gamma\to\omega_0$. For $\gamma>\omega_0$ we need a different form for the Hamiltonian K to preserve its Hermitian nature. We transform away the first part of the expression given in equation (5.3), rather than the second part. Thus we let

$$\begin{pmatrix}A\\ A^{\dagger}\end{pmatrix}\to\begin{pmatrix}C\\ C^{\dagger}\end{pmatrix}=\begin{pmatrix}\cosh v & -\mathrm{i}\sinh v\\ \mathrm{i}\sinh v & \cosh v\end{pmatrix}\begin{pmatrix}A\\ A^{\dagger}\end{pmatrix}, \tag{6.9}$$

with

$$\cosh v=[(\gamma+\eta)/(2\eta)]^{1/2},\qquad \sinh v=[(\gamma-\eta)/(2\eta)]^{1/2}. \tag{6.10}$$

Then, corresponding to equations (6.4) and (6.5), we find

$$K\to K''=-\tfrac{1}{2}\mathrm{i}\hbar\eta(C^{\dagger 2}-C^2),\qquad [C,C^{\dagger}]=1, \tag{6.11}$$

$$H\to H''=\hbar\omega_0(\gamma/\eta)(C^{\dagger}C+\tfrac{1}{2})+\tfrac{1}{2}\mathrm{i}\hbar(\omega_0^2/\omega)(C^{\dagger 2}-C). \tag{6.12}$$

The time dependence of the operator C is found easily from its equation of motion $\mathrm{d}C/\mathrm{d}t=(\mathrm{i}\hbar)^{-1}[C,K'']$. We obtain

$$C(t)=C(0)\cosh\eta t-C^{\dagger}(0)\sinh\eta t, \tag{6.13}$$

from which it may be checked that K given by equation (6.11) is time-independent.

Let us apply the transformation (6.9) a second time:

$$\begin{pmatrix}\bar{C}\\ \bar{C}^{\dagger}\end{pmatrix}=\begin{pmatrix}\cosh 2v & -\mathrm{i}\sinh 2v\\ \mathrm{i}\sinh 2v & \cosh 2v\end{pmatrix}\begin{pmatrix}A\\ A^{\dagger}\end{pmatrix}=\frac{1}{\eta}\begin{pmatrix}\gamma & -\mathrm{i}\omega_0\\ \mathrm{i}\omega_0 & \gamma\end{pmatrix}\begin{pmatrix}A\\ A^{\dagger}\end{pmatrix}. \tag{6.14}$$

Corresponding to equation (6.8) we find

$$K\to\bar{K}(\omega_0\to-\omega_0)=-\hbar\omega_0(\bar{C}^{\dagger}\bar{C}+\tfrac{1}{2})-\tfrac{1}{2}\mathrm{i}\hbar\gamma(\bar{C}^{\dagger 2}-\bar{C}^2),\qquad [\bar{C},\bar{C}^{\dagger}]=1. \tag{6.15}$$

Comparing with equation (5.3), we see that this corresponds to the obvious symmetry $\omega_0 \to -\omega_0$. Of course both symmetries $\gamma \to -\gamma$, $\omega_0 \to -\omega_0$ apply irrespective of the strength of the damping. Putting $\eta \to -i\omega$ in (6.14), $\omega \to i\eta$ in (6.7) gives the corresponding transformation in the undercritical and overcritical cases respectively.

We shall continue our investigation of the case of overcritical damping in the next section.

7. Connection with the Schrödinger representation

We find that the matrix elements given by equation (3.8) may be obtained from an operator

$$H_s = \hbar(\omega_0^2/\omega)(D^\dagger D + \tfrac{1}{2}) + \tfrac{1}{2}i\hbar\gamma[(1 - i\gamma/\omega)D^{\dagger 2} - (1 + i\gamma/\omega)D^2], \tag{7.1}$$

where

$$[D, D^\dagger] = 1, \qquad D^\dagger D|m\rangle = m|m\rangle, \qquad m = 0, 1, 2, \ldots. \tag{7.2}$$

We can check our working in §§ 3 and 6.1 by seeking a canonical transformation

$$H_s \to \bar{H} = U^\dagger H_s U, \qquad \langle n|\bar{H}|n'\rangle = \hbar\omega_0(n + \tfrac{1}{2})\delta_{nn'} \tag{7.3}$$

that diagonalises H_s to a form in agreement with equation (6.1). By writing the transformation in the form

$$\begin{pmatrix} \bar{D} \\ \bar{D}^\dagger \end{pmatrix} = \begin{pmatrix} e^{i\phi}\cosh w & -i\sinh w \\ i\sinh w & e^{-i\phi}\cosh w \end{pmatrix} \begin{pmatrix} D \\ D^\dagger \end{pmatrix}, \tag{7.4}$$

subject to $[\bar{D}, \bar{D}^\dagger] = 1$, we find

$$\bar{H} = \hbar\omega_0(\bar{D}^\dagger \bar{D} + \tfrac{1}{2}), \qquad \bar{D}^\dagger \bar{D}|n\rangle = n|n\rangle, \qquad n = 0, 1, 2, \ldots \tag{7.5}$$

where

$$\begin{pmatrix} \bar{D} \\ \bar{D}^\dagger \end{pmatrix} = \left(\frac{\omega(\omega + \omega_0)}{2\omega_0^2}\right)^{1/2} \begin{pmatrix} 1 + i\gamma/\omega & i\gamma/\omega \\ -i\gamma/\omega & 1 - i\gamma/\omega \end{pmatrix} \begin{pmatrix} D \\ D^\dagger \end{pmatrix}. \tag{7.6}$$

We return now to the case of heavy damping. The Schrödinger equation is easily solved in this case. The Hamiltonian is given by equation (6.11) and the eigenvalue equation becomes

$$i\hbar\eta(Q\,d/dQ + \tfrac{1}{2})\psi_\lambda(Q) = \lambda\psi_\lambda(Q). \tag{7.7}$$

The spectrum is continuous, the system being similar to a repulsive oscillator (Leach 1980). Corresponding to an eigenvalue λ, $-\infty < \lambda < \infty$, the eigenfunction is

$$\psi(Q, \lambda) = (2\pi\eta\hbar)^{-1/2} Q^{-1/2 - i\lambda/(\eta\hbar)}, \tag{7.8}$$

with the usual continuous-spectrum normalisation

$$\int_{-\infty}^{\infty} \psi^*(Q, \lambda)\psi(Q, \lambda')\,dQ = \delta(\lambda - \lambda'). \tag{7.9}$$

This completes the wave-mechanical solution of the decaying oscillator which we started in § 3.

8. The Green function

The Green function $G(Q, Q_0, t)$ is the solution of the Schrödinger equation with initial condition

$$G(Q, Q_0, 0) = \delta(Q - Q_0). \tag{8.1}$$

We follow Sargent *et al* (1974) by writing this in the form

$$Q(-t)G(Q, Q_0, t) = Q_0\delta(Q - Q_0). \tag{8.2}$$

8.1. Case of undercritical damping

We work in the Heisenberg representation and insert our result (4.3a) into equation (8.2), taking care to change the sign of γ for $t<0$. This leads to

$$(1/G)\partial G/\partial Q = \mathrm{i}M_0\omega(\hbar \sin \omega t)^{-1}\{[\cos \omega t - (\gamma/\omega)\sin \omega t]Q - Q_0\} \tag{8.3}$$

with solution

$$G(Q, Q_0, t) = (M_0\omega/2\pi\hbar|\sin \omega t|)^{1/2} \exp\left\{\frac{\mathrm{i}M_0}{2\hbar \sin \omega t}\right.$$
$$\left.\times\left[(Q^2 + Q_0^2)\left(\cos \omega t - \frac{\gamma}{\omega}\sin \omega t\right) - 2QQ_0\right]\right\}. \tag{8.4}$$

8.2. Case of overcritical damping

All we have to do to obtain the result corresponding to equation (8.4) in the case $\gamma > \omega_0$ is to make the analytic continuation $\omega \rightarrow \mathrm{i}\eta$.

9. Coherent states

9.1. Case of undercritical damping

From our reduction of the problem to equations (6.4), it is clear that we may use the number states of the operator $B^\dagger B$ to construct coherent states of the lightly decaying oscillator in exactly the same way as for an undamped oscillator. Thus for any complex eigenvalue β of the operator B we may construct

$$|\beta, t\rangle = \exp(-\tfrac{1}{2}|\beta|^2) \sum_{l=0}^{\infty} \beta^l \exp(-\mathrm{i}l\omega t)(l!)^{-1/2}|l\rangle, \tag{9.1a}$$

where

$$B^\dagger B|l\rangle = l|l\rangle, \qquad l = 0, 1, 2, \ldots. \tag{9.1b}$$

9.2. Case of overcritical damping

As is to be expected, the situation is quite different in the case of overcritical damping. We can no longer construct states that will cohere for any length of time. We content ourselves by examining the disintegration of states of the undamped oscillator (1.5) formed at time $t = 0$. It is most convenient to start with a coherent state of the operator

$C^\dagger C$ at time $t=0$:

$$|\alpha\rangle = \exp(-\tfrac{1}{2}|\alpha|^2) \sum_{n=0}^{\infty} \alpha^n (n!)^{-1/2} |n\rangle, \qquad (9.2a)$$

where

$$C^\dagger C|n\rangle = n|n\rangle, \qquad n = 0, 1, 2, \ldots. \qquad (9.2b)$$

When overcritical damping has acted for a time t this state evolves to

$$|\alpha, t\rangle = U(t)|\alpha\rangle, \qquad (9.3)$$

where equation (6.11) gives the evolution operator

$$U(t) = \exp[-\tfrac{1}{2}\eta t(C^{\dagger 2} - C^2)]. \qquad (9.4)$$

Using the method of Munn and Silbey (1978), we can factorise the operator to give

$$U(t) = \exp(\tfrac{1}{2}C^{\dagger 2} \tanh \eta t) \exp[-(C^\dagger C + \tfrac{1}{2}) \ln \cosh \eta t] \times \exp(-\tfrac{1}{2}C^2 \tanh \eta t). \qquad (9.5)$$

Substituting into equation (9.3) and using $C|\alpha\rangle = \alpha|\alpha\rangle$, $\exp(\lambda C^\dagger C)|\alpha\rangle = \exp(|\alpha|^2 \mathrm{e}^\lambda \sinh \lambda)|\alpha\, \mathrm{e}^\lambda\rangle$, we obtain

$$|\alpha, t\rangle = (\operatorname{sech} \eta t)^{1/2} \exp(-\tfrac{1}{2}\alpha^2 \tanh \eta t) \exp(\tfrac{1}{2}|\alpha|^2 \tanh^2 \eta t) \times \exp(\tfrac{1}{2}C^{\dagger 2} \tanh \eta t)|\alpha \operatorname{sech} \eta t\rangle. \qquad (9.6)$$

As the disintegration is of most interest when it first sets in, we consider the case when $t \ll 1$; then to first order in ηt

$$|\alpha, t\rangle = (1 - \tfrac{1}{2}\eta t\alpha^2 + \tfrac{1}{2}\eta t C^{\dagger 2})|\alpha\rangle. \qquad (9.7)$$

The disintegration will occur mainly through the presence of the operator $C^\dagger$ which acts according to the equation

$$C^\dagger|\alpha\rangle = (\partial/\partial\alpha + \tfrac{1}{2}\alpha^*)|\alpha\rangle. \qquad (9.8)$$

10. Conclusion

A canonical Hamiltonian K has been derived which closely resembles a time-dependent integral of the motion found by Dodonov and Man'ko (1979). However, the Hamiltonian K is time-independent (since $(1/M)\,\mathrm{d}M/\mathrm{d}t$ is constant) and hence it possesses a stationary system of eigenvalues and eigenvectors. This enables us to analyse the quantal motion of a damped harmonic oscillator in a way that is straightforward and reliable, depending as it does on well established procedures for time-independent Hamiltonians.

An important feature of the system is the transition at critical damping $\gamma = \omega_0$ from an equivalent undamped harmonic oscillator of frequency $\omega = (\omega_0^2 - \gamma^2)^{1/2}$, in the case $0 < \gamma < \omega_0$, to a continuous-spectrum system, akin to a repulsive oscillator (Leach 1980), in the case $\gamma > \omega_0$. The behaviour of coherent states shows the essentially different natures of the system in these two cases. For $0 < \gamma < \omega_0$, coherent states survive indefinitely, just as in the case $\gamma = 0$, but for $\gamma > \omega_0$, owing to the effective repulsion, such states have zero lifetime.

We have already made some comparisons with the work of other authors. In § 3 we remarked on the factor $\exp(-\tfrac{1}{2}\gamma t)$ in the wavefunctions of Hasse (1975) and of Tartaglia

(1977). We have changed the sign of γ to fit in with our notation; in the reports of the above authors the factor actually gives a growth with time which is difficult to understand. This factor is seen to be absent from our result (3.5), the decay of the system being represented only in the transformed coordinate $Q = q \exp(-\gamma t)$ which occurs in the wavefunction (3.5). Normalisation is thus independent of the time. The analysis in § 5, being based on M_0 rather than on $M(t)$, does not contain the exponential growth and decay factors which are present at all stages in the work of Hasse (1975) and of Dodonov and Man'ko (1979). For instance, the latter authors obtain results for the coordinate $q(t) = Q(t) \exp(\gamma t)$ and the momentum $p(t) = P(t) \exp(-\gamma t)$ which contain both of the factors $\exp(\pm\gamma t)$, whereas our expressions (4.3a, b) give $q(t) \propto \exp(\gamma t)$, $p(t) \propto \exp(-\gamma t)$.

As we remarked in the Introduction, we have taken $\gamma > 0$ because the physics of a cavity oscillator demands this choice. For the damped harmonic oscillator problem treated in isolation the sign of γ may be changed, so that the growth and decay of q, p can be interchanged at will. This is an aspect of the uncertainty principle: if we observe $q(t)$, then we see a decay; similarly if we observe $p(t)$ we see a decay. In any case the relation $\Delta Q \Delta P \geqslant \frac{1}{2}$, which follows from equations (1.5), implies $\Delta q \Delta p \geqslant \hbar/2$. This resolves some earlier difficulties connected with the uncertainty principle (see e.g. Tartaglia 1977).

A most important quantity is the energy $E(t)$ stored in the system. Taking the cavity oscillator discussed in the Introduction, the energy is given by equation (1.6) and it is interesting to examine its behaviour for large values of t. In the case of undercritical damping $(0 < \gamma < \omega_0)$ H is given by adding equations (4.10) and (4.11), and shows oscillatory behaviour. Taking a mean value of H, we may write for large values of t

$$E(t) \approx \exp(-2\gamma t) E(0). \tag{10.1}$$

For overcritical damping, $\gamma > \omega_0$, we see from equation (4.16) that

$$E(t) \sim \exp[-2(\gamma - \eta)t] E(0), \qquad \eta > 0. \tag{10.2}$$

Let us consider two special cases of overcritical damping:

(a) $\gamma \gtrsim \omega_0 \Rightarrow \eta \approx 0$

$$\Rightarrow E(t) \sim \exp(-2\gamma t) E_c(0), \qquad E_c(0) \gg K;$$

(b) $\gamma \gg \omega_0 \Rightarrow \eta \approx \gamma - \varepsilon \qquad (\varepsilon \gtrsim 0)$

$$\Rightarrow E(t) \sim \exp(-2\varepsilon t) E_\infty(0), \qquad E_\infty(0) \approx T(0) + V(0).$$

In case (a) the system exerts only a slight repulsion and energy can be put in easily, but decays rapidly. In case (b) we have used equation (4.16) to examine the behaviour of $E_\infty(0)$. We expect $T(0)$, $V(0)$ and K to be of comparable magnitude, so that $E_\infty(0) \ll E_c(0)$. In the latter case the system is essentially a 'brick wall': not much energy can be put into the system, but this small amount decays slowly.

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